Boise Math Circle
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Spook-tacular Mental Math!

by Gary Thomas (expanded 2016 version)

Ghosts don't have bodies, so they can't hold on to pencils, which means they have to do all their math in their heads. So here it comes: mental math!

A few **comments** before we begin:

- Mental math is not really "math", it's just arithmetic
- Just because you can do mental math doesn't mean you're smart
- Anyone can do mental math, even if they don't know the algebra behind these methods
- Playing with numbers in your head turns them into very familiar friends
- And mental math is a boatload of fun!

By the way, these aren't "secrets" in the sense that you can't share them. In fact, the best way to get good at something (besides lots of practice) is to teach it to someone else. By participating in today's session, you are agreeing to teach some of these great mental math techniques to a friend.

Preliminary Skills

You need to know the **times table** up to 12 x 12.

Doubling numbers is easy. The trick is to realize there is never a carry greater than 1, so you can double left to right without much effort by "peeking" just a little bit at the next digit.

Double these numbers: 2 7 12 16 19 37 89 458 461 1,028 1,892 6,210

Halving numbers (dividing by 2) is also easy. The trick is to try to group the digits so they make an even number, which is lots easier to halve than an odd number is.

Halve these numbers: 2 7 12 16 19 37 89 458 461 1,028 1,892 6,210

Subtracting from 9 is ridiculously simple, but have some patience, we will build on it.

Subtract these numbers from 9:   3   5   8

Subtracting from 10 is also ridiculously simple, but we will build on it, too.

Subtract these numbers from 10:   3   5   8

Forming the "**one's complement**" of a number (this terminology comes from computer science) just means to subtract each digit of the number (from left to right) from 9, and then subtract the one's place digit from 10.

Form the one's complement of these numbers: 7   12   617   10,945

Subtracting from the next highest power of 10 is now a piece of cake. It's just the one's complement.
Perform these subtractions: $10 - 7$ $100 - 12$ $1{,}000 - 617$ $100{,}000 - 10{,}945$

Surprising Techniques That Only Ghosts Know

Multiplying by Dividing
If you want to multiply by 5, you can multiply by 10 (just add a 0 on the end) and then divide by 2 – this is the halving that you practiced a few minutes ago.

Multiply by 5:   2   7   12   16   19   37   89   458   461   1,028   1,892   6,210

Dividing by Multiplying
If you want to divide by 5, you can double the number then divide it by 10 (move the decimal point over one place to the left).

Divide by 5: 2 7 12 16 19 37 89 458 461 1,028 1,892 6,210

Adding by Subtracting
Some subtraction problems are complicated because of tricky borrowing. Instead, add the next highest round number and then subtract to adjust for the overage. Suppose we want to add 38 and 75. This could get messy, but we notice that 38 is almost 40, so we say to ourselves, "38, almost 40, plus 75 is 115, but that's 2 too many, so 113."

Add: $108 + 76$ $319 + 47$ $681 + 598$ $8{,}355 + 1{,}487$

Subtracting by Adding
If you want to subtract a number, add its one's complement to a number below it.
Here's an example.

Suppose we want to subtract 89 from 635. This is kind of an annoying problem, because it has all kinds of borrowing and silly nonsense like that to trip us up. So we say to ourselves, "89 – one's complement is 11, 635 becomes 535, add the 11, we get 546." This is WAY easier!

If you work with bigger numbers, just count down a bigger multiple of the correct power of 10. Suppose we want to subtract 656 from 2819. We say to ourselves, "656, that's almost 700, and the one's complement would be 44. So 2,819 minus the 700 is 2,119, plus the 44 is 2,163." (And notice that to add the 19 and the 44 I used the "adding by subtracting" trick. I added 20 to 44 and got 64, then subtracted 1 to get 63.)

Subtract: $84 - 7$ $612 - 58$ $2839 - 477$

Special Purpose Tricks That Don't Always Work, But When They Do They're Great

Multiplying by 11
Many of you might already know this trick, because it's a favorite of all spooky creatures. If you have several digits in your number, then you have to move from right-to-left (which we almost never do in mental math), but with practice you can still do it from left to right, and with only 2 digits, just add the two digits together and put the result between the two original digits – if you have a carry, take care of it.

Multiply by 11: 63 38 29 421 1,065

Squaring a Number That Ends in 5
This is another one that most of you probably already know. Just add one to whatever comes before the 5, multiply it, then put 25 at the end. If we want to square 85, we say to ourselves, "85, 8 plus 1 is 9, 8 times 9 is 72, put 25 at the end, so 7225." Easy-peasy. Ask Gary later if you want to see the algebra for why this works.

Square these numbers: 15 35 45 75 105 145

Was that last one kind of hard? OK, here is a special purpose trick to make it easier.

Multiplying Numbers in the Teens
This is one of my favorite tricks. When I was your age, I of course had to learn the multiplication table up to 12 times 12. But with this trick, you will know the multiplication table up to 19 times 19! (Actually, no, you won't – but you will be able to multiply in your head so fast that everyone will think you have memorized it that high.)

To use this trick, add the one's digit of one of the numbers to the other number. Multiply by 10 (just add a zero, or just shift it left in your mind), then multiply both one's digits together and add. Suppose we want to multiply 13 times 19. We say to ourselves, "Hmm, numbers in the teens. Add the 3 from the 13 to the 19, get 22, shift to 220, 3 times 9 is 27, and 220 plus 27 is 247."

Multiply: 12×14 13×18 17×14 19×19

Now square these numbers: 115 125 135 145 155 165 175 185 195

Multiplying by 9

Multiplying by 9 is just multiplying by 10 and then subtracting once. Suppose we want to multiply 56 by 9. We say to ourselves, "56, times 10 is 560, minus 56 is 504." If the subtracting is more difficult, just add the one's complement. Suppose we want to multiply 87 by 9. We say to ourselves, "87, times 10 is 870, adjust to 770, add 13 to get 783." I decide whether to just subtract or to use the one's complement depending on whether the one's digit is greater than the ten's digit.

Multiply: 8×9 12×9 34×9 43×9 88×9

Multiplying by 99
Multiplying by 99 is actually even easier than multiplying by 9, because it is multiplying by 100 and then subtracting once. Since you are subtracting, you will have to adjust the hundreds by one, and then just say the one's complement. Suppose we want to multiply 56 by 99. We would say to ourselves, "56, one less is 55, one's complement of 56 is 44, put those together to get 5,544."

Multiply: 8×99 12×99 34×99 43×99 88×99

"2x2" Multiplication

The rest of the methods in today's math circle will focus on multiplying two-digit by two-digit numbers. These are still easy to do, but are quite impressive to people who you haven't taught yet.

Multiplying by Factors
If you need to multiply a number that is easily factored, you can break the problem down into pieces. Suppose we want to multiply 28 times 41. We might say to ourselves, "Hmmm, 28 is 4 times 7, so 7 times 41 is 287, to multiply by 4 we double twice, so 574, then 1,148."

Multiply: 21×14 25×86 48×27

Using an Anchor
This is a more advanced technique, but very useful when the two numbers are close to each other. Decide on an anchor number that is easy to multiply (usually a multiple of 10). Determine how far off the two numbers are from the anchor. Add one of those values to the other whole number and then multiply by the anchor. Then multiply the two differences and add (or subtract) from that result. A few examples should clear this up. Suppose we want to multiply 42 by 47. We say to ourselves, "Hmmm, 42 and 47 are both close to 40, and multiplying by 4 is easy, so let's pick 40 as the anchor. One number is 2 more than 40, the other is 7 more than 40. Let's add

the 2 to the 47 to get 49. Multiply by 4 (just double it twice, 98, 196), to get 196, and we are actually multiplying by 40, so 1,960. Now the 2 and the 7 are multiplied to give us 14 and we add that to 1,960 to get 1974."

Here's another one, a little harder. Suppose we want to multiply 83 by 86. We say to ourselves, "Hmmm, 83 by 86, they are both close to 90, and multiplying by 9 is easy, so let's pick 90 as the anchor. One number is 7 less than 90, the other is 4 less than 90. Let's reduce 86 by 7 to get 79, then multiply by the anchor 90. To multiply by 9, tack a 0 on the end of 79 to get 790, then subtract 79 to get 711, and of course we are multiplying by 90, so it's 7,110. Now the differences were -7 and -4, and their product is 28, so add 28 to 7,110 to get 7,138."

If one number is above the anchor and one number is below, then you subtract the final product. Suppose we want to multiply 58 by 65. We say to ourselves, "Hmmm, 58 by 65, they are both close to 60, so let's pick 60 as the anchor. One number is 2 less than 60, the other is 5 more than 60. Let's reduce 65 by 2 to get 63, then multiply by the anchor 6. Here I multiply 60 by 6 for 360, then 3 by 6 for 18, add to get 378, then adjust because we are multiply by 60, not 6, and get 3,780. Now one number was 2 less, one was 5 more, so multiply 2 by 5 to get 10, and subtract from 3,780 to get 3,770."

Remember, if both numbers are above your anchor, add the final product. If both numbers are below your anchor, add the final product. But if one number is higher and one number is lower, then you *subtract* the final product.

Multiply: 21×28 59×61 75×83

Special Anchor Value of 100

If the two numbers are close to 100, use 100 as the anchor. Multiplying by 100 as the anchor is so easy a baby ghost could do it. Suppose we want to multiply 92 by 96. We say to ourselves, "Hmmm, 92 by 96, both are close to 100, so let's pick 100 as the anchor (our fave!). One number is 8 less than 100, the other is 4 less than 100. Let's reduce 92 by 4 to get 88, and multiplying by our anchor of 100 just gives us 8,800, no brain-power required. Now, the differences were -8 and -4, and their product is 32, so add 32 to 8,800 to get 8,832." These problems are especially fun to do in your head, because they are extra easy but because the numbers are so big, others think they are extra hard, so they are very impressive to do. Make sure you teach others the secrets so that they can have fun with these, too!

Multiply: 91×94 93×98 92×95

Special Anchor Value of 50

If the two numbers are close to 50, use 50 as the anchor. Multiplying by 50 is the same as multiplying by 5 (halving) with an extra 0 or decimal point shift at the end. Suppose we want to multiply 52 by 54. We say to ourselves, "Hmmm, 52 by 54, both are close to 50, so let's pick 50 as the anchor. One number is 2 more than 50, the other is 4 more than 50. Let's increase 54 by 2 to get 56, and multiplying by our anchor of 50 gives us 2,800 (half of 56 is 28, then add the 00). Now the differences were 2 and 4, and their product is 8, so add 8 to 2,800 to get 2,808."

Multiply: 48×44 53×56 58×61 47×55

Memorizing the Squares

If you can memorize the perfect squares from 1^2 to 100^2, there are lots of other great tricks. In this section, we will get you started, but you will have to keep working on this on your own to memorize all of them. First of all, here are all hundred squares for you to look at.

1	1	11	121	21	441	31	961	41	1681	51	2601	61	3721	71	5041	81	6561	91	8281
2	4	12	144	22	484	32	1024	42	1764	52	2704	62	3844	72	5184	82	6724	92	8464
3	9	13	169	23	529	33	1089	43	1849	53	2809	63	3969	73	5329	83	6889	93	8649
4	16	14	196	24	576	34	1156	44	1936	54	2916	64	4096	74	5476	84	7056	94	8836
5	25	15	225	25	625	35	1225	45	2025	55	3025	65	4225	75	5625	85	7225	95	9025
6	36	16	256	26	676	36	1296	46	2116	56	3136	66	4356	76	5776	86	7396	96	9216
7	49	17	289	27	729	37	1369	47	2209	57	3249	67	4489	77	5929	87	7569	97	9409
8	64	18	324	28	784	38	1444	48	2304	58	3364	68	4624	78	6084	88	7744	98	9604
9	81	19	361	29	841	39	1521	49	2401	59	3481	69	4761	79	6241	89	7921	99	9801
10	100	20	400	30	900	40	1600	50	2500	60	3600	70	4900	80	6400	90	8100	100	10000

Work in pairs, to be awares, of patterns you notice, in the squares (yes, Gary's a poet):

It will take you some time to memorize all the squares, but you can do it if you practice. There are "only" 100 of them, and you've already noticed some patterns. Plus, you can use the "teen multiplication rule" to get the squares up to 20^2, and the "close to 100" rule for the squares in the eighties and nineties. Gary will also show you the trick for squares from 40^2 to 60^2 in the next little section. This means that even without thinking about it, you already know 40 of the 100 squares, and you also know the squares of numbers ending in 5, so you have memorized almost half of them (or at least you can calculate them so fast that it's as good as memorizing them). Use the patterns to gradually learn the rest. Gary practices them every morning in the shower. Can you recite all of them in the time it takes you to suds up and rinse off? Say them out loud and your entire family will eventually know them.

Squares from 40 to 60

This is an absolutely marvelous trick, a favorite of all disembodied beings. To use this trick, you have to know that $50^2 = 2{,}500$, which is pretty easy, and you also have to know the squares from 1 to 9 (also pretty easy). Then when you have to square a number between 40 and 60, ask yourself how far it is from 50. If it's below 50, subtract that much from 25. If it's above, add that much to 25. Then tack on the square of the difference from 50. For example, to square 42, first we notice that 42 is 8 below 50, so in our mind we think "$25 - 8 = 17$" to get our hundreds, then we know that $8^2 = 64$, so we tack that on and say "42^2 is 1764". And it's really that easy. Squaring 51 would involve noticing that 51 is 1 more than 50, so we add 1 to 25 and think of 26 for our hundreds, then tack on $1^2 = 01$ (we need two digits), and say "51^2 is 2601".

Now you try it, squaring the following: 57 46 53 59 44

If you don't mind adding a bit more, you can extend this from 30 to 70, because you have memorized (or know the trick) to calculate squares in the teens. For example, to square 68, we notice that 68 is 18 more than 50, so we add 18 to 25 and get 43 initially for our hundreds. Then we use the teens trick to square 18 and get 324, then add that to 4,300 to get 4,624 as the square of 68.

Square the following: 62 39

Squaring Any Old Number

Memory is the best for squaring, so practice as often as you can, but if you forget one, you can still square any 2-digit number in your head, even if it isn't one of our special cases. What you do is square the first digit to get the hundreds, square the last digit to get the tens and ones, and then multiply the two digits together and double to get more tens. For example, if we couldn't remember the square of 38, we say "3 squared is 9, so 900, 8 squared is 64, so 964, now multiply 3 by 8 to get 24, double it to get 48, that makes 480, added to 964 is 1,444." This will take a bit of practice, but is very good for you brain, as any spooky spirit will tell you if you just ask.

In my mind, I think of "square the first, square the last, multiply and double". Try these using that method:

27 63 77

Depending on the time, Gary will show you some other squaring tricks, or you could ask him about them later if you're just dying to know them.

Difference of Squares

So now that you've memorized all the squares, you can use the difference of squares technique, which is very useful for two numbers that are reasonably close to each other. It works a lot better if both numbers are odd, or both numbers are even. Suppose we want to multiply 72 and 76. These numbers are close to each other, and they are both exactly 2 away from 74, which is the middle of them. You know from memory that 74^2 is 5,476, and then you subtract the square of that distance from the middle value of 74. So in math terms, it would look like $72 \times 76 = (74 - 2) \times (74 + 2) = 74^2 - 2^2 = 5{,}476 - 4 = 5{,}472$. That probably looks like a lot of work, but you will be surprised how easy it becomes with some practice. The more squares that you have memorized, the more you can use this technique.

Multiply: 28×32 31×39 46×48 77×83

Keep practicing. Whenever I am in the car, I look at license plates and multiply the numbers on them together. You can start by taking just three digits, and turning those into a 1-digit times 2-digit problem. Practice like that for a while, and then practice 2-digit by 2-digit. You can also take two digits at a time and practice your perfect squares.

Sources

Speed Mathematics: Secret Skills for Quick Calculation, by Bill Handley

This is a good book for beginners, and includes not only the techniques in today's Math Circle but also more tricks for multiplication, as well as techniques for fractions and square roots.

Speed Math for Kids: The Fast, Fun Way to Do Basic Calculations, by Bill Handley

This has most of the material in Handley's other book, but is written at a simpler level. Any sixth-grader in the Math Circle would be able to read this book and master its tricks. If you have a little brother or sister at home, you can teach them what's in this book and you'll both become experts at mental math.

Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks, by Arthur Benjamin and Michael Shermer

This book includes everything in Handley's books and a fair amount more, but is written at a slightly more advanced level. It would be readable by any high school student.

The Great Mental Calculators: The Psychology, Methods, and Lives of Calculating Prodigies Past and Present, by Steven B. Smith

This book has a lot of history about the fascinating men and women (and boys and girls!) who were experts in mental math. It also describes many of the techniques as well.

Dead Reckoning: Calculating without Instruments, by Ronald W. Doerfler

This book is one of my favorites, but it is very advanced. It includes techniques for mentally calculating not only the usual multiplication and square roots, but also factoring, trigonometric functions, and exponents and logarithms! Doerfler also has a website at **http://www.myreckonings.com/Dead_Reckoning/Dead_Reckoning.htm.**

The Trachtenberg Speed System of Basic Mathematics, by Rudolph McShane and Jakow Trachtenberg

Trachtenberg's book was a favorite of mine when I was younger, so I mention it here, but I find it less useful as I read the other books on this list. If you like the "11" trick, this book has lots more tricks like that, but I feel as though they get so complicated that it's easier to just multiply in my head than to use the trick. Trachtenberg's story is interesting in itself. He was a Jewish mathematician who was imprisoned by the Germans during World War II, and developed his system in order to have something to do with his mind so he wouldn't just go crazy.