Summing infinitely many numbers

Adding a bunch of numbers. You are probably familiar with how to add a lot of numbers together. For example to calculate

$$1 + 6 + 19 + 33 = ?$$

You would first add 1 and 6 to get 7, then add 19 to get 26, and finally add 33 to get 59.

Adding infinitely many numbers and getting infinity. But what happens if you try to add infinitely many numbers together? Sometimes nothing interesting happens. For example try to calculate the summation

$$1 + 1 + 1 + 1 + ... = ?$$

Here we mean that there are 1's being added forever. If you try to add them all, you get infinity!

Adding infinitely many numbers together and getting a number. Now we will see that sometimes good things can happen. For example, we will try to calculate

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots = \ ?$$

Using a calculator, fill in the following table.

The first term =	
The sum of the first two terms =	
The sum of the first three terms =	
The sum of the first four terms =	
The sum of the first five terms =	
The sum of the first six terms =	
The sum of the first seven terms =	
...	
...	
... (do as **many** as it takes!)	
...	
...	
My guess for the value of the summation =	

In this example you may have noticed a pattern in the values in the table. When you find patterns like that, you can save time even compared with a calculator!

Some examples

For each of the following infinite sums, use the **table method** to add the first two terms, first three terms, first four terms, etc. Then

- Look for patterns in the values of the table
- Describe what is happening in the long run
- Decide whether the Sum has a value or not

(a) $1 + .5 + 1 + .5 + 1 + .5 + \cdots =$

(b) $1 - .1 + 1 - .1 + 1 - .1 +- \cdots =$

(c) $1 - 1 + 1 - 1 + 1 - 1 +- \cdots =$

(d) $1 + .1 + .01 + .001 + .0001 + .00001 + \cdots =$

(e) $3 + .1 + .04 + .001 + .0005 + .00009 + \cdots =$
(we are cheating a little here :)

(f) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots =$

Infinite summations with a ratio

Let's look again at example (d), where the terms are generated by a common **ratio** of 1/10. Each term is exactly 1/10 of the previous term! This is similar to the classic **Zeno's paradox**: Suppose you have to go a distance of 2 miles. In the first step you go 1 mile, then 1/2 mile, then 1/4 mile, and so on. With infinitely many of these to accomplish, can you ever reach your goal of 2 miles?

Try the following examples of summations with a ratio.

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots =$
(this is Zeno!)

(b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots =$
(now Zeno goes a little more slowly :)

(c) $1 + .9 + .81 + .729 + \cdots =$
(what is the ratio here?)

(d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + - \cdots =$
(what is the ratio here?)

(e) $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \cdots =$
(what is the ratio here?)

What kind of patterns do you see in your guesses for these summations?

Common ratio summations and algebra

Let's study the summation $1 + r + r^2 + r^3 + \cdots$ using algebra. The key idea is to look at what happens to the summation when you multiply it by r.

- Start with the equation

$$S = 1 + r + r^2 + r^3 + \cdots$$

- Multiply both sides by r. On the left-hand side you get:

$$r \cdot S =$$

Write in the right-hang side.

- Now substract the second equation from the first. On the left-hand side you get:

$$S - r \cdot S =$$

What do you get on the right-hand side? Be sure to get rid of all the terms that cancel.

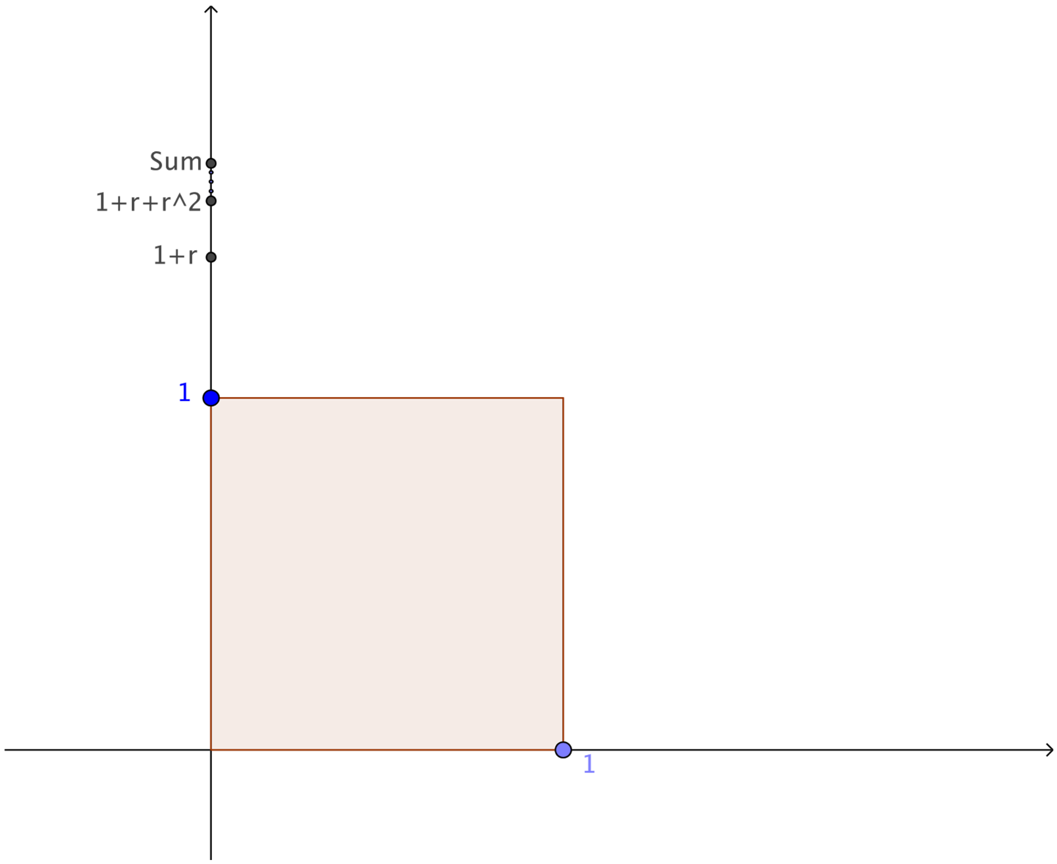
- Now use the previous part to solve for the sum *S*!

- Bonus: Can you use the method above to "prove" the following?

$$1 + 2 + 4 + 8 + 16 + \cdots \;=\; -1$$

Common ratio summations and geometry

Let's study the summation $1 + r + r^2 + r^3 + \cdots$ using geometry. On the y-axis we have placed a segment of length 1, followed by a segment of length r, followed by a segment of length r^2, and so on. The segments end at the point A which is the summation we are looking for. We have also drawn a unit box.



- Draw the diagonal line from "Sum" to the 1 on the x-axis. Where does this line meet the top of the square?

- The diagonal line creates a big triangle with base 1 and height Sum. But it also creates a smaller triangle with base B and height H. What are B and H?

- The similar triangle rule says that

$$\frac{1}{Sum} = \frac{B}{H}$$

  Can you use this to solve for the Sum?

Infinite summations of powers

Sometimes a summation doesn't have a constant ratio between its terms, but does have another simple kind of pattern.

Try the following examples of summations with a ratio.

(a) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots =$

(b) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots =$

(c) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots =$

(d) $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots =$

Which of the above summations have a value? Which ones are infinity?

Bonus: Write down the value of the sum in (a), multiply it by 6, and take the square root of that. What do you get?

The summation of 1/n

Revisit problem (c) from the previous page.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots = \ ?$$

You might have guessed that this summation has a value somewhere between four and five. In fact this summation tends very **slowly** to **infinity**!

To see why this is the case, show that each of the following equations is true.

- $\frac{1}{3} + \frac{1}{4} \geq \frac{1}{2}$

- $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{2}$

- Write down the next one and explain why it is true.

- Why does this pattern mean that the given summation is infinity?

- About how many terms might you need to sum to show the summation surpasses 20?