Patterns in Two-Coin Systems[i]
Boise Math Teachers' Circle
October 2016

Imagine that we introduce a new coin system. Instead of using pennies, nickels, dimes and quarters, let's agree on using 4-cent and 7-cent coins. You probably noticed the following flaw of this new system: certain amounts cannot be made, for example, 1, 2 or 5 cents. On the other hand this makes the new system more interesting! So let's investigate.

1. What amounts can be made with a two-coin system made of 4-cent and 7-cent coins?

    a. After you are reasonably confident about which amounts can be made, take a moment to reflect on the results. What surprised you? What observations did you make along the way?

    b. Is there a largest amount that cannot be made in this system? If so, what is it?

2. What do you think will happen if we play the same game with the numbers 5 and 11—that is, what amounts can be made with 5-cent and 11 cent coins? Do you think there will be a biggest amount that cannot be made? Make a prediction and then determine an answer.

   a. Make an argument in support of your claim about the biggest amount that cannot be made in the 5-cent, 11-cent system.

   b. Then, try to convince your neighbors.

3. Now, take a look at the number of amounts that cannot be made in the two systems you have considered thus far.

   a. How many coins cannot be made in the 4-cent, 7-cent system?

   b. How many coins cannot be made in the 5-cent, 11-cent system?

   c. What interesting pattern can you find between the amounts that cannot be made and the largest amount that cannot be made? You might try considering the number of amounts that CAN be made among the ones that cannot. Make a conjecture based on your pattern and be prepared to share with neighbors.

Notice that we have been exploring two-coin systems where the coin values do not share a common factor. In general, let's name our coins a and b. Wouldn't it be nice to have a formula for the largest amount that cannot be made with the coins a and b?

4. Let's break away from the coin context for a moment to solve a few equations. First, find a pair of integers x and y such that $4x + 7y = 1$. Can you find another such pair? What about another? What interesting things are you noticing?

5. Document your observations by making a general claim about the solutions to $4x + 7y = 1$. Be prepared to justify your claim.

6. Now, find a solution to the equation $4x + 7y = 17$.

7. Find a solution to $4x + 7y = 10{,}025$. If you haven't already tried this, try using one of your solutions to $4x + 7y = 1$.

8. Show that for any integer t, there exists integers x and y such that $4x + 7y = t$.

9. What about for $ax + by = t$ where a and b are coins that do not share a common factor—will you always be guaranteed a solution? Explain.

Working our way back to coins (which are positive integers)…

10. Now, try forcing an added restriction on the number x. If we stipulate that $0 \leq x \leq 6$, can we always find integers x and y such that $4x + 7y = t$, for a given integer t?

11. Let's consider the general case with coins a and b where the coins do not share a common factor. If we restrict x so that $0 \leq x \leq b - 1$, can we find a solution to $ax + by = t$ for a given positive integer t?

12. Show the following algorithm works for determining whether or not a given amount t can be made (we will use the coins 4 and 7):
    a. Start with the positive integer t.
    b. Find integers x and y such that $4x + 7y = t$ and $0 \leq x \leq 6$. (You know you can by part 10.)
    c. The only instances where t can be made from the coins occurs precisely when $y \geq 0$.

13. State the algorithm above for the coins a and b.

14. Now use your work with the algorithm to determine a way to compute the largest amount that cannot be made in the 4-cent, 7-cent system. Generalize your formula for any two coins a and b where the coins do not share a common factor.

---

[i] Ideas for this problem set came directly from the Oakland/East Bay Teachers' Circle handout titled, "Coins, M&Ms, and Generating Functions," which was found online at http://www.mathteacherscircle.org/resources/math-sessions/.