Dimension and self-similar geometry

Samuel Coskey and Zach Teitler, Boise State University

A simple idea of dimension (work in your break-out groups)

1) Let S be a line segment. If you scale S by an integer factor N, how many copies of the original segment S are needed to cover the scaled segment?

2) Let R be a rectangle. If you scale R by an integer factor N in both directions, how many copies of the original rectangle R are needed to cover the scaled rectangle?

3) Answer the same question for a rectangular box B (the box has a width, length, and height).

4) Can you write down an equation relating the scaling factor, the number of copies, and the dimension?

5) All these shapes were "self-covering"—the scaled shape can be covered by copies of the original shape. Is a disk self-covering? A ball?

Cantor set activity (work together as a class)

The Cantor set is a subset of the real line which is constructed in steps:

- Step 0: begin with the number line segment from 0 to 1
- Step 1: delete the (open) middle third (leave 2 segments of length 1/3 each)
- Step 2: delete the middle third of each segment
- Step n+1: delete the middle thirds of each segment from step n
- The Cantor set is the "limit" (intersection) of this process

1. Draw the first few steps of the process.

2. What is the dimension of the Cantor set?

Koch snowflake (group 1 work on this)

The Koch snowflake is constructed in several steps.

- Step 0: begin with the segment from (0,0) to (1,0)

- Step 1: replace the middle third of the segment with two line segments of length 1/3 to make a “tent” (so now there are 4 segments of length 1/3)
- Step 2: for each of the 4 segments in Step 1, replace the middle third with two line segments to make a “tent”. (Now there are 16 segments of length 1/9.)
- Step n+1: replace the middle third of each segment from step n with a tent
- The Koch snowflake is the “limit” of this process

1. Draw the first few steps of the process.

2. What is the dimension of the Koch snowflake?

3. (Bonus question) How long is the Koch snowflake?

Sierpinski gasket (group 2 work on this)

The Sierpinski gasket is constructed in several steps.

- Step 0: begin with a solid equilateral triangle
- Step 1: connect the midpoints of the sides of the triangle, thus dividing it into 4 triangular pieces. Delete the triangular piece in the middle.
- Step 2: delete the “middle triangle” from each of the three remaining triangles
- Step n+1: delete the middle triangle from each remaining triangles from step n

1. Draw the first few steps of the process.

2. What is the dimension of the Sierpinski gasket?

3. (Bonus question) What is the area of the Sierpinski gasket?

A tree-like fractal (group 3 work on this)

This tree fractal is constructed in several steps.

- Step 0: begin with the segment from (0,0) to (0,1). this will be the “trunk”.
- Step 1: attach two branches of length 2/3 to the top of the trunk. Use angles of roughly 70° and 110° (they don’t have to be exact).
- Step 2: attach a new “child” branch of length 4/9 to the tip of each of the two braches above. Use the same sort of angles, but relative to the “parent” branch.
- Step n+1: attach a new child branch to the tip of each of the 2^n branches from step n. Make them 2/3 as long as the parent branch, and use a similar angle spread.

1. Draw the first few steps of the process.

2. What is the dimension of the tree?

3. (Bonus question) According to Wikipedia, a cauliflower is similar but each branch has about 13 child branches of about 1/3 its length. What is the dimension of cauliflower?

Crumpled paper ball (all groups work on this)

In this activity we will estimate the dimension of a crumpled paper ball!

- Crumple a sheet of paper into a ball. Crumple it very **firmly** and make it as **round** as you can. Measure the diameter of the ball as accurately as you can.
- Cut a sheet in two and crumple one of the halves into a ball. Measure the diameter of this smaller ball.
- Compare your results with those of your group. If necessary, keep working until you can agree on a reasonably consistent set of measurements.

1. Thinking of your formula from the previous activities, what value serves as the number of copies here?

2. What value serves as the scaling factor?

3. Based on your measurements, what is the dimension of your crumples?

Further thoughts

1. What about non self-similar shapes? Say N(r) is how many size r boxes/disks are needed to cover the shape; we predict N(r) $\approx$ (constant)$\cdot r^d$, where the constant depends on the overall shape (and on the choice of boxes or disks), and d is the dimension. Try finding N(r), and d, for
   - a single point
   - a finite set
   - a segment
   - a circle
   - a rectangle
   - a filled-in disk

   This is called the Minkowski dimension (there is also the much more famous Hausdorff dimension which is the same for "most" cases, and more technical to define, yet mentioned much more often).

2. What is a **fractal** really? One definition of a fractal is any set whose Hausdorff dimension (or Minkowski dimension, or counting-scaled-copies dimension) is different than its topological dimension. Unfortunately we have not had time to discuss topological dimension, but (a) it is usually the intuitive thing: points are 0, segments are 1, rectangles are 2, etc., and (b) it is always integer-valued. So if a set's Hausdorff dimension is anything other than an integer, then the set is definitely a fractal. And even if the dimension is an integer, it could still be a fractal if it's a different integer than the topological dimension. And we've seen that self-similar sets are a great way to build up fractals (but not the only way).

3. Look up some pictures of fractals on the internet!

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A simple idea of dimension (work in your break-out groups)

For the following 3 questions, use the table below.

1) Let S be a line segment. If you scale S by an integer factor N, how many copies of the original segment S are needed to cover the scaled segment?
2) Let R be a rectangle. If you scale R by an integer factor N in both directions, how many copies of the original rectangle R are needed to cover the scaled rectangle?
3) Answer the same question for a rectangular box B (the box has a width, length, and height).

	Draw a picture	scaling factor	# of copies	dimension
Line Segment				
Rectangle				
Rectangular Box				

4) Can you write down an equation relating the scaling factor, the number of copies, and the dimension?

5) All these shapes were “self-covering”—the scaled shape can be covered by copies of the original shape. Is a disk self-covering? A ball?

Cantor set activity (work together as a class)

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- The Cantor set is the “limit” (intersection) of this process

1. Draw the first few steps of the process.

2. Scale your set by 9. Look at the number of copies between this scaled version version the original segment between 0 to 1.

3. What is the dimension of the Cantor set?

4. Do we need to use a scaling factor of 9? What if we calculate the dimension of moving between the 0 to 1 segment and the 0 to ⅓ segment?

Koch snowflake (group 1 work on this)

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3. (Bonus question) What is the area of the Sierpinski gasket?

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- Measure the diameter of the ball as accurately as you can.
  Compare your results with those of your group. If necessary, keep working until you can agree on a reasonably consistent set of measurements.

  Agreed group measurement of the diameter of the ball: ____________ (cm)

- Cut a sheet in two
- Crumple one of the halves into a ball.
- Measure the diameter of the smaller ball as accurately as you can.
  Compare your results with those of your group. If necessary, keep working until you can agree on a reasonably consistent set of measurements.

  Agreed group measurement of the diameter of this smaller ball: ____________ (cm)

- Thinking of your formula from the previous activities.

1. What is the number of copies for this activity?

2. What value serves as the scaling factor?

3. Based on your measurements, what is the dimension of your crumples?

Further thoughts

1. What about non self-similar shapes?
   Say N(r) is how many size r boxes/disks are needed to cover the shape;
   we predict N(r) $\approx$ (constant)$\cdot r^d$, where the constant depends on the overall shape (and on the choice of boxes or disks), and d is the dimension.

   Try finding N(r) and d for the following shapes:

	Picture	N(r)	d
a single point			
a finite set			
a segment			
a circle			
a rectangle			
a filled-in disk			

   This is called the **Minkowski dimension** (there is also the much more famous Hausdorff dimension which is the same for "most" cases, and more technical to define, yet mentioned much more often).

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References

Mandletbrot Set https://www.youtube.com/watch?v=gEw8xpb1aRA

Notes for kids:

- Start with solid shapes: segment, rectangle, box
- Do solid triangle too
- Then do Sierpinski triangle (argue it should be between 1 and 2)
- Now Koch snowflake on their own
- (do the full snowflake, it just involves some cutting)
- Crumple ball
- discuss 3d figures (sponges)
- Any between 0 and 1: Cantor type sets (optional) (or separate lesson?)