Irrational and rational numbers

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1. Here are some decimals. Do they represent rational numbers? Can you write them as fractions with integer numerator and denominator?

(a) $x = 0.\overline{72}$ meaning $0.727272\ldots$ with 72 repeating

(b) $y = 3.14159\overline{26}$

(c) $z = 3.1415$

2. How would you explain why every real number with a repeating decimal, or eventually repeating decimal, is rational?

3. Find the decimal expansions of these rational numbers. Are they repeating? To gain insight use long division (not a calculator).

(a) $a = 1/7$

(b) $b = 1/13$

(c) $c = 1/130$

(d) $d = 2/13$

4. Why do rational numbers have repeating (or eventually repeating) decimals?

5. (Digressions.)

- Find the decimal expansions of 1/1, 1/2, 1/3,... up to 1/20. What are the *periods* of the repeating decimals? How does the period of the decimal of $1/n$, compare to n?
- If n is any positive integer with no factors of 2 or 5, then n has an integer multiple whose digits are all 9. For example, 7 has the multiple 999999 (six 9s) because 7 times 142857 is 999999. Does 13 have a multiple whose digits are all 9? Can you explain why this works for any n (and why it doesn't work if n has factors of 2 or 5)?

⋆ We've seen reasons why repeating decimals represent rational numbers, and rational numbers have repeating decimals. Now let's explore some irrational numbers.

6. $x = 0.101001000100001000001 0\ldots$, given by a decimal with just 0s and 1s, with the 1s separated by $1, 2, 3, 4, \ldots$ zeros.

Is x rational or irrational? How do you know?

7. $\sqrt{2}$

- Can you tell by looking at decimals of $\sqrt{2}$ on a calculator or computer? Why or why not?
- What if we set $(a/b)^2 = 2$, in other words $a^2 = 2b^2$? Then would a and b be even or odd? What would happen?
- Do you think $\sqrt{2}$ is rational or irrational? Can you explain why it is?

8. (Additional numbers that we can show are irrational.)

- $\sqrt{3}$
- $\sqrt[3]{2}$
- $x = \log_2(3)$

9. Here is a proof that e is irrational.

(a) As a reminder, e^x has the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We are interested in $x = -1$:

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

(Dropping the $1 - 1$ at the beginning, that came from $1 + x$.)

(b) With a calculator find the sums of the first 1, 2, 3, 4 terms of this. Observe

- The partial sums go up, down, up, down, up, down...
- The jumps are getting smaller.
- The "high" partial sums are decreasing, and the "low" partial sums are increasing. From a "high" partial sum, the jump down and then back up doesn't get all the way back up. From a "low" partial sum, the jump up and then down doesn't get all the way back down.

- The partial sums alternate between being over $1/e$ and under $1/e$. Every step jumps across $1/e$, back and forth.

From this, we can see that when we add up N terms, the "error" (distance from $1/e$) has to be less than the magnitude of the next, $(N+1)$'st term. Otherwise we'd be too far away for that $(N+1)$'st term to make its jump across.

(c) On the other hand the "error" can never equal 0—in an alternating, decreasing series like this, the partial sums are never equal to the infinite ("final") sum. If this happened, then the $(N+1)$'st term would jump away from the value, but that would put us too far away for the $(N+2)$'nd term to jump back across.

(d) So if we add up N terms of the right hand side, we know it is within $1/(N+1)!$ of $1/e$, but not equal to $1/e$. Rewrite this as an absolute value inequality for the absolute value of the difference between the N'th partial sum, and $1/e$.

(e) Now say $e = a/b$, so $1/e = b/a$. And say $N > a$, so $(N+1)!$ is divisible by a. Clear denominators by multiplying through by $(N+1)!$. We get $0 < |\text{something}| < 1$ where the item in the middle is some integer. This is impossible, so $e = a/b$ couldn't have happened in the first place.

⋆ Decimals are not the only way to represent real numbers. Another way is called "continued fractions". General continued fractions look like

$$a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\epsilon}{f + \dots}}}}}$$

We'll focus on *simple continued fractions*, where the numerators are always 1 and the numbers in the denominators are integers ≥ 0.

10. What is

$$x = 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{9}}}}}$$

Is this x rational or irrational? How do you know?

11. What about

$$y = 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{8 + \cfrac{1}{1}}}}}}$$

12. Rewrite as simple continued fractions:

- 8/5
- 13/8?
- 314/271, and 271/314

13. x is rational if and only if the continued fraction of x...

14. What about

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$

Is x rational or irrational? How can you tell? Is this x a recognizable (familiar) number—can it be rewritten in a more familiar form?

15. What about

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}}$$

16. What about

$$x = 3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{3 + \cdots}}}}}}$$

17. What kind of numbers seem to have "periodic" continued fractions?

18. We can find a continued fraction for numbers such as $x = \sqrt{2}$. There are several ways, including using a calculator. We know $x = 1.4142135624\ldots$ so

$$x = 1 + 0.4142135624\ldots$$

which we can rewrite as

$$x = 1 + \frac{1}{\left(\frac{1}{0.4142135624\ldots}\right)}$$

And we can rewrite that as

$$x = 1 + \frac{1}{2.4142135624\ldots}$$

or in other words

$$x = 1 + \frac{1}{2 + 0.4142135624\ldots}$$

Using a calculator, continue this development a few more steps. What does the continued fraction of $\sqrt{2}$ look like?

19. (Could we do the same thing without a calculator, by rationalizing numerators?)

20. How about π?

21. Try "cutting off" a continued fraction at some point and simplifying it. For example, in

$$y = 3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{9}}}}}$$

we could form the “cut-offs”

$$3, \qquad 3+\frac{1}{1}, \qquad 3+\cfrac{1}{1+\cfrac{1}{4}}, \qquad 3+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1}}}, \qquad 3+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{5}}}}, \qquad \dots$$

How close are they to y? What if we try similar things for $\sqrt{2}$ or π?

22. What does the continued fraction of e look like?

23. (A digression.) Some other continued fraction expansions (technically, not simple, but perhaps you will feel that non-technically they are simple):

$$e-1 = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\cfrac{5}{6+\cfrac{6}{7+\dots}}}}}}}$$

and

$$e = 1+\cfrac{2}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cdots}}}}}}$$

and

$$e^{x/y} = 1+\cfrac{2x}{2y-x+\cfrac{x^2}{6y+\cfrac{x^2}{10y+\cfrac{x^2}{14y+\cfrac{x^2}{18y+\cfrac{x^2}{22y+\cdots}}}}}}$$

For π,

$$\frac{4}{\pi} = 1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \cfrac{25}{11 + \cfrac{36}{13 + \cdots}}}}}}$$

This one was discovered in 1999:

$$\pi = 3 + \cfrac{1}{6 + \cfrac{9}{6 + \cfrac{25}{6 + \cfrac{49}{6 + \cfrac{81}{6 + \cfrac{121}{6 + \cdots}}}}}}$$

Here's one that was discovered in 2008:

$$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \cfrac{1}{\frac{1}{5} + \cfrac{1}{\frac{1}{6} + \cfrac{1}{\frac{1}{7} + \cdots}}}}}}}$$

Wow!

Proving that these are actually correct statements takes some work.

24. Is $e+\pi$ rational or irrational? This is an *open question*. Mathematicians believe that $e+\pi$ is irrational. (Partly because, c'mon, look at it; and there are some related conjectures and theorems that suggest this should be a theorem, too.) But no proof is known; and who knows, perhaps it's rational!