Introduction to Combinatorics

Combinatorics (think combinations) is a branch of math focused mainly on ways to list objects (**enumeration**) and determine how many (**counting**). It also includes studying ways to create finite structures with certain properties (**design**), such as networks and graphs.

This activity introduces four types of counting problems.

Counting Problem #1.
4 songs are on your device: {Aura, Believe, Candle, Dandy}
You want to choose 2 songs for a playlist.
You may choose the same song twice, and it matters to you which song comes first.

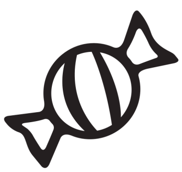
List all the playlists! How many are there?
AA
AB
BA

Counting Problem #2.
4 kinds of fruit are available: {Apple, Banana, Clementine, Drupe}
People are asked to choose their favorite and their second favorite.

List all the possibilities! How many are there?

Counting Problem #3.
5 candidates run for student council: {Abacus, Bobathy, Cawdor, Dalia, Elephant}
2 of the candidates will be elected.

List all the possibilities! How many are there?



Counting Problem #4.
4 pieces of (identical) candy will be given to 3 people: {Abacus, Bobathy, Cawdor}
Not everyone needs a piece of candy, and the sharing doesn't need to be fair.

List all the possibilities! How many are there?

Sampling Problems

A sampling problem asks how many ways there are to choose objects from a set. Different kinds of rules can change the answer.

Issue 1. When you pick an object, do you put it back or keep it out?

- If you put it back, you are **sampling with replacement**.
- If you keep it out, you are **sampling without replacement**.

Issue 2. Do you care about the order in which you pick the objects?

- If you care about which objects are picked first, second, etc, then **order matters**.
- If you only care about which objects are picked, then **order doesn't matter**.

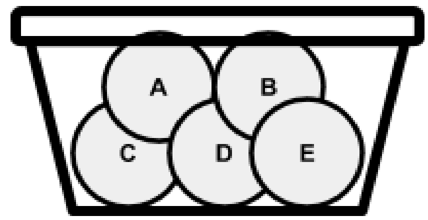
The Counting Problems are examples of sampling problems, but with different rules. Think about each one carefully and fill in the table below.

	with replacement or **without replacement**?	**order matters** or **order doesn't matter**?
Counting Problem #1		
Counting Problem #2		
Counting Problem #3		
Counting Problem #4		

The four possibilities each relate to different kind of **mathematical structure**:

Sampling with replacement, order matters	*sequence*
Sampling without replacement, order matters	*sequence of distinct elements*
Sampling without replacement, order doesn't matter	*set*
Sampling with replacement, order doesn't matter	*multiset*

Sampling with Replacement, Order Matters



You want to sample from a basket with 5 balls: {A,B,C,D,E}

You pick one at a time **with replacement** (putting it back each time), and the **order matters** to you.

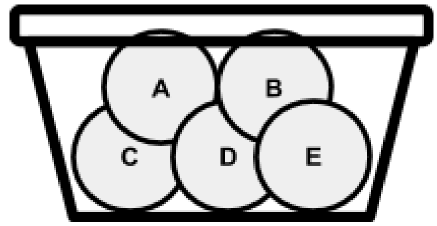
- How many ways are there to select 1 ball?
- How many ways are there to select 2 balls?
- How many ways are there to select 3 balls?
- How many ways are there to select 4 balls?

What pattern do you see?

Question. With these rules, if you have ***N*** balls, how many ways are there to select ***K*** objects?

Does this help confirm your answer to Counting Problem #1?

Sampling without Replacement, Order Matters



You want to sample from a basket with 5 balls: {A,B,C,D,E}

You pick one at a time **without replacement** (keep what you pick), and **order matters** to you.

- How many ways are there to select 1 ball?
- How many ways are there to select 2 balls?
- How many ways are there to select 3 balls?
- How many ways are there to select 4 balls?

What pattern do you observe?

Question. With these rules, if you have ***N*** balls, how many ways are there to select ***K*** objects?

It may be helpful to know about **factorial**, which is written using an exclamation point.

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

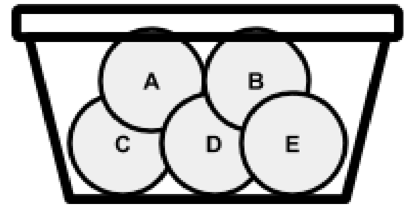
$4! = 4 \times 3 \times 2 \times 1 = 24$

...

$N! = N \times (N-1) \times \cdots \times 2 \times 1$

Bonus: Can you rewrite your answer to the above Question using ***N***, ***K***, and factorials?

Sampling without Replacement, Order Doesn't Matter



You want to sample from a basket with 5 balls: {A,B,C,D,E}

You pick them one at a time **without replacement** (keep what you pick), and **order doesn't matter** to you.

- How many ways are there to select 1 ball?

- How many ways are there to select 2 balls?

- How many ways are there to select 3 balls?

- How many ways are there to select 4 balls?

What pattern do you observe?

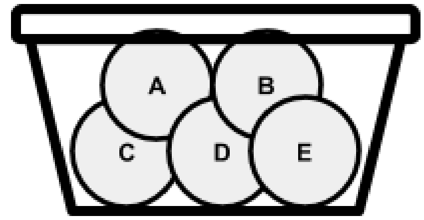
Question. With these rules, if you sample from ***N*** balls, how many ways are there to select ***K*** objects? Feel free to explain your ideas in words.

> The answer to this question is so important that mathematicians have given it a special symbol: $\binom{N}{K}$ or $_{N}C_{K}$ on a calculator. It is pronounced "***N* Choose *K***".

Bonus: Can you write $\binom{N}{K}$ using ***N***, ***K***, and factorials?

Sampling with Replacement, Order Doesn't Matter

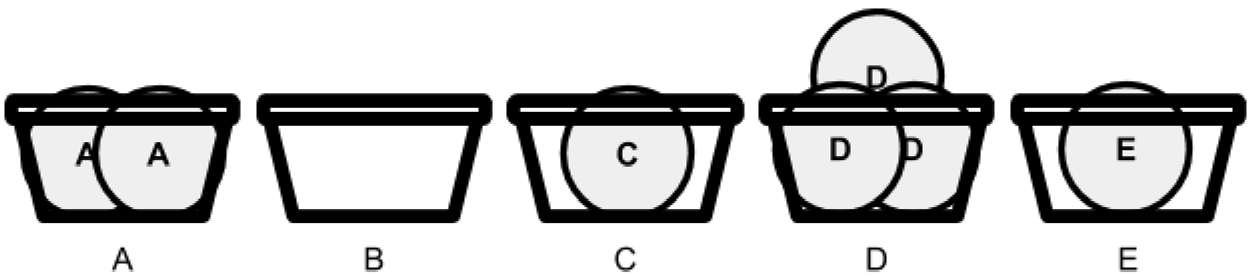
You want to sample from a basket with 5 balls: {A,B,C,D,E}



You select them one at a time **with replacement** (putting it back each time), and **order doesn't matter** to you.

Before trying to count the possibilities, it helps to know a good way to write them. The trick is to think of putting the balls you pick into baskets, each containing zero or more balls.

For example, if you sample 7 items, you could organize the balls you pick using 5 baskets:



We can "code" this sample using symbols for the balls and a separator symbol:

O O | | O | O O O | O

This requires a total of 11 symbols: 7 balls and 4 separators.

Discuss this [as a group or class].

Question. With these rules, If you sample from ***N*** balls, select ***K*** items, and code it this way, how many symbols must you use?

Big bonus: With these rules, if you sample from ***N*** balls, how many ways are there to select ***K*** objects? Write the answer using the Choose idea from before.