Dimension and self-similar geometry

Samuel Coskey, Boise State University

A simple idea of dimension (work in your break-out groups)

For the following 3 questions, use the table below.

1. Let S be a line segment. If you scale S by an integer factor N, how many copies of the original segment S are needed to cover the scaled segment?
2. Let R be a rectangle. If you scale R by an integer factor N in both directions, how many copies of the original rectangle R are needed to cover the scaled rectangle?
3. Answer the same question for a rectangular box B (the box has a width, length, and height).

	Draw a picture	scaling factor	# of copies	dimension
Line Segment				
Rectangle				
Rectangular Box				

4. Can you write down an equation relating the scaling factor, the number of copies, and the dimension?

5. All these shapes were “self-covering”—the scaled shape can be covered by copies of the original shape. Is a triangle self-covering? A disk?

Activity 1 - Cantor set (work together as a class)

The Cantor set is a subset of the real line which is constructed in steps:

- Step 0: begin with the number line segment from 0 to 1
- Step 1: delete the (open) middle third (leave 2 segments of length 1/3 each)
- Step 2: delete the middle third of each segment
- Step n+1: delete the middle thirds of each segment from step n
- The Cantor set is the “limit” (intersection) of this process

1. Draw the first few steps of the process.

2. Scale your set by 9. Look at the number of copies between this scaled version version the original segment between 0 to 1.

3. What is the dimension of the Cantor set?

4. Do we need to use a scaling factor of 9? What if we calculate the dimension of moving between the 0 to 1 segment and the 0 to ⅓ segment?

Activity 2 - Koch snowflake (work at your tables)

The Koch snowflake is constructed in several steps.

- Step 0: begin with the segment from (0,0) to (1,0)
- Step 1: replace the middle third of the segment with two line segments of length 1/3 to make a “tent” (so now there are 4 segments of length 1/3)
- Step 2: for each of the 4 segments in Step 1, replace the middle third with two line segments to make a “tent”. (Now there are 16 segments of length 1/9.)
- Step n+1: replace the middle third of each segment from step n with a tent
- The Koch snowflake is the “limit” of this process

1. Draw the first few steps of the process.

2. What is the dimension of the Koch snowflake?

3. (Bonus question) How long is the Koch snowflake?

Activity 3 - Crumpled paper

In this activity we will estimate the dimension of a crumpled paper ball!

- Crumple one big sheet of paper into a ball. Crumple it very **firmly** and make it as **round** as you can.
- Measure the diameter of the ball as accurately as you can.
  Compare your results with those of your group. If necessary, repeat the experiment until you can agree on a reasonably consistent set of measurements.

  Agreed group measurement of the diameter of the ball: ____________ (cm)

- Put two big sheets of paper together on an edge to make a double wide sheet.
- Crumple the double-wide sheet into a ball.
- Measure the diameter of the lager ball as accurately as you can.
  Compare your results with those of your group. If necessary, keep working until you can agree on a reasonably consistent set of measurements.

  Agreed group measurement of the diameter of the larger ball: ____________ (cm)

- Thinking of your formula from the previous activities.

1. What value serves as the “number of copies” for this activity?

2. What value serves as the “scaling factor” for this activity?

3. Based on your measurements, what is the dimension of crumpled paper? Does the value you get seem reasonable?

Further thoughts (take home)

1. What about non self-similar shapes? In that case we let N(r) be the number of balls of radius r that are needed to cover the shape. We predict N(r) $\approx$ (constant)$\cdot r^d$, where the constant depends on the overall shape, and d is the dimension.

   This is called the **Minkowski dimension** (there is the more powerful and oft-mentioned Hausdorff dimension—in “most” cases the two have the same value, but Hausdorff dimension is more technical to define).

2. What is a **fractal** really? One definition of a fractal would be a set whose Minkowski dimension is different than its topological dimension. Unfortunately we have not had time to discuss topological dimension, but (a) it is usually the intuitive thing: points are 0, segments are 1, rectangles are 2, etc., and (b) it is always a whole number. So in particular if a shape has Minkowski dimension that is not a whole number, then we would consider it to be a fractal. Self-similarity is a great way to make fractals, but not the only way.

3. Look up some pictures of fractals on the internet!
   https://www.youtube.com/watch?v=gEw8xpb1aRA